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Visualization of Effect Algebras by Automorphisms

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We show that every effect algebra satisfying the Riesz decomposition property can be represented as an effect algebra of automorphisms of an antilattice, and every MV-algebra can be represented as an MV-algebra of automorphisms of a linearly ordered set. Such a representation enables us to visualize effect algebras by functions. This is a variation of the Holland representation theorem for ℓ-groups and of its generalization of Glass for directed interpolation po-groups as ℓ-groups or po-groups automorphisms of linearly ordered set or of an antilattice, respectively.

KEY WORDS: effect algebra; MV-algebra; antilattice; prime ideal; automorphism; unital po-group; unital ℓ-group.

1. INTRODUCTION

Effect algebras entered mathematics and, in particular, quantum structures at the beginning of nineties by the Foulis and Bennett (1994) motivating by D-posets (Kôpka and Chovanec, 1994), and they combine both algebraic and fuzzy ideas that are involved in the most important example, the set of effect operators, $\mathcal{E}(H)$, the set of all Hermitian operators of a Hilbert space H that are between the zero and identity operator.

Nowadays effect algebras are an important part of quantum structures that are a mathematical background of quantum mechanics. In many cases, such algebras are intervals in po-groups, like $\mathcal{E}(H)$ is an interval in the po-group $\mathcal{B}(H)$, the set of all Hermitian operators of a Hilbert space H.

Ravindran (1996) showed that every effect algebra satisfying the Riesz decomposition property (RDP) is also an interval in an interpolation Abelian unital po-group. Similarly, MV-algebras are also intervals in Abelian unital ℓ-group as it follows from the famous result of Mundici (1986). Recently, Riečanová showed every lattice effect algebra can be covered by blocks, maximal compatible sets, which are in fact MV-algebras. So that, when quantum logics and Boolean algebras correspond to measurement in quantum mechanics or Newtonian mechanics,

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respectively, so do effect algebras and MV-algebras when our measured data are roughly speaking unsharp or fuzzified.

In the present paper we show that every effect algebra satisfying (RDP) and every MV-algebra can be visualized, i.e., represented by the effect algebra and MV-algebra of automorphisms of an antilattice or of a linearly ordered set. This gives an analogue of the famous Holland representation theorem (Holland, 1963) of ℓ-groups as a set of automorphisms of a linearly ordered set, as well as its generalization of Glass (1972) for directed interpolation po-groups.

The paper is organized as follows. In the Section 2 we define effect algebras and their representation. Section 3 deals with ideals and mainly with prime ideals showing that an ideal is prime iff its quotient is an antilattice. The main representation results are in the Section 4. Having such a representation, we are able to visualized and better to describe the structure of effect algebras.

2. EFFECT ALGEBRAS

An *effect algebra* is by Foulis and Bennett (1994) a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in$ E,

(i) $a + b$ is defined in E iff $b + a$ is defined, and in such the case $a + b = b + a$;
(ii) $a + b$, $(a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and in such the case $(a + b) + c = a + (b + c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;
(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$.

For example, if (G, u) is an Abelian unital po-group with a strong unit u,[2] and if

$$\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$$

is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

Let E and F be two effect algebras. A mapping $h : E \to F$ is said to be a *homomorphism* if (i) $h(a + b) = h(a) + h(b)$ whenever $a + b$ is defined in E, and (ii) $h(1) = 1$. A bijective homomorphism h such that h^{-1} is homomorphism is said to be an *isomorphism* of E and F.

[2] An element $u \in G^+$ is said to be a *strong unit* for a po-group G, if given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property*, (RIP) for short, if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_i$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_i$ for all i, j; (ii) the *Riesz decomposition property*, (RDP) for short, if $x \leq y_1 + y_2$ implies that there exist two elements $x_1, x_2 \in$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice, then E has trivially the (RIP); the converse is not true as wee see below. (2) E has (RDP) iff (Dvurečenskij and Pulmannová, 2000, Lemma 1.7.5), $x_1 + x_2 = y_1 + y_2$ implies there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, and $y_2 = c_{12} + c_{22}$. (3) (RDP) implies the (RIP), but the converse is not true (e.g., if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without (RDP). On the other hand, every finite poset with the (RIP) is a lattice.

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have a supremum or infimum. It is clear that any linearly ordered poset is an antilattice and every finite effect algebra with the (RIP) is a lattice.

A partially ordered Abelian group $(G; +, 0)$ is said to satisfy the *Riesz decomposition property* provided, given x, y_1, y_2 in G^+ such that $x \leq y_1 + y_2$, there exist x_1, x_2 in G^+ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for each j. This condition is equivalent by Goodearl (1986, Proposition 2.1) with the following two equivalent conditions:

(a) Given x_1, x_2, y_1, y_2 in G such that $x_i \leq y_j$ for all i, j, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j.
(b) Given x_1, x_2, y_1, y_2 in G^+ such that $x_1 + x_2 = y_1 + y_2$, there exist $z_{11}, z_{12}, z_{21}, z_{22}$ in G^+ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

According to Goodearl (1986), such a group G with is said to be the *interpolation group*.

Ravindran (1996) (Dvurečenskij and Pulmannová, 2000, Theorem 1.17.17) proved the following important result.

Theorem 2.1. *Let E be an effect algebra with (*RDP*). Then there exists a unique (up to isomorphism of unital Abelian po-groups) unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic with E.*

Remark 2.2. We recall that in Theorem 2.1 all finite meets and joins from E are preserved in (G, u), see Dvurečenskij and Vetterlein (2001, Proposition 6.3). Moreover, there is also a categorical equivalence among the category of effect algebras with (RDP) and the category of interpolation Abelian unital po-groups (Dvurečenskij, submitted).

A very important family of effect algebras are MV-algebras that entered mathematics by Chang (1958).

We recall that an MV-*algebra* is an algebra $M := (M; \oplus, \odot, ^*, 0, 1)$ of type (2,2,1,0,0) such that, for all $a, b, c \in M$, we have

(MVi) $a \oplus b = b \oplus a$;
(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(MViii) $a \oplus 0 = a$;
(MViv) $a \oplus 1 = 1$;
(MVv) $(a^*)^* = a$;
(MVvi) $a \oplus a^*$=1;
(MVvii) $0^* = 1$;
(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

If we define a partial operation $+$ on M in such a way that $a + b$ is defined in E iff $a \leq b^*$, then $a + b := a \oplus b$, the $(M; +, 0, 1)$ is an effect algebra. According to Mundici (1986), every MV-algebra is isomorphic to the set $\Gamma(G, u)$, where $a + b = (a + b) \wedge u$, and $a^* = u - a (a, b \in \Gamma(G, u))$.

MV-algebras have appeared in the realm of effect algebras in many natural ways: Mundici (1986) showed that starting from an AF C*-algebras we can obtain countable MV-algebras, and conversely, any countable MV-algebra can be derived in such a way. Bennett and Foulis (1995) introduced Φ-symmetric effect algebras that are exactly MV-algebras, and also Boolean D-posets of Chovanec and Kôpka (1992) are MV-algebras.

MV-algebras play a similar role in effect algebras as Boolean algebras do in orthomodular posets—they describe maximal sets of mutually (strongly) compatible elements. Moreover, Riečanová (2000) recently proved an important result that each lattice ordered effect algebra can be covered by MV-subalgebras that form blocks.

3. IDEALS AND QUOTIENTS OF EFFECT ALGEBRAS

An *ideal* of an effect algebra E is a nonempty subset I of E such that (i) $x \in E, y \in I, x \leq y$ imply $x \in I$ and (ii) if $x, y \in I$, and $x + y$ is defined in E, then $x + y \in I$. An ideal I is said to be the *Riesz ideal* if $x \in I, a, b \in E$ and $x \leq a + b$, there exist $a_1, b_1 \in I$ such that $x \leq a_1 + b_1$ and $a_1 \leq a$ and $b_1 \leq b$.

For example, if E is with (RDP), then any ideal of E is Riesz.

We denote by $\mathcal{I}(E)$ the set of all ideals of E; then $\{0\}, E \in \mathcal{I}(E)$.

The ideal $I_0(a)$ of E fulfilling (RDP) generated by an element a is the set

$$I_0(a) = \left\{x \in E : \exists\, k \in \mathbb{N}, \exists\, a_1^0, \ldots, a_k^0 \in E, a_i^0 \leq a, x = a_1^0 + \cdots + a_k^0\right\}.$$

We say that an ideal P of an effect algebra E with (RDP) is *prime* if, for all ideals I and J of E, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(E)$ the set of all prime ideals of E.

As it was proved in Dvurečenskij (submitted), if a is a nonzero element of an effect algebra E with (RDP), there is an ideal I that is maximal under the condition $a \notin I$; such an ideal is always prime (Dvurečenskij, submitted, Proposition 3.5); if $a = 1$, any such ideal is said to be *maximal*, and let $\mathcal{M}(E)$ denote the set of all maximal ideals of E.

Let P be an ideal of an effect algebra E with (RDP). We define a relation $\sim_P$ on E via $a \sim_P b$ iff $a - e = b - f$ for some, $e, f \in P$. Then $\sim_P$ is a relation on E, and the set $E/P := \{a/P : a \in E\}$, where $a/P := \{b \in E : b \sim P_a\}$, can be converted into an effect algebra $(E/P, +, 0/P, 1/P)$ such that $a/P + b/P = c/P$ if and only if there are $a_1 \in a/P$, $b_1 \in b/P$, and $c_1 \in c/P$ such that $a_1 + b_1 = c_1$. Moreover, if E has (RDP), so has E/P.

The following important characterization of prime ideals was proved in Dvurečenskij (submitted, Proposition 6.5).

Proposition 3.1. *A proper ideal P of E with (*RDP*) is prime if and only if E/P is an antilattice.*

We recall that an *o-ideal* of a po-group G is any directed convex subgroup of G. An o-ideal I of a po-group G is said to be (i) *maximal* if it is a proper subset of G and it is not contained in any proper o-ideal of G, (ii) *prime* if, for all o-ideals P and Q of G with $P \cap Q \subseteq I$, we have $P \subseteq I$ or $J \subseteq I$, and (iii) a *value* of a nonzero element g if $g \notin I$ and it is maximal with respect to this property. Let (G, u) be a unital Abelian pogroup; by $\mathcal{I}(G)$, $\mathcal{M}(G)$, and $\mathcal{P}(G)$ we denote the set of all o-ideals, maximal o-ideals, and prime o-ideals, respectively, of G.

The following result from Dvurečenskij (submitted) gives a one-to-one connection among the sets of all ideals, maximal and prime ideals and the sets of o-ideals, maximal o-ideals, and prime o-ideals.

Theorem 3.2. *Let (G, u) be a unital interpolation po-group and let $E = \Gamma(G, u)$. For any ideal I of E, we assign*

$$\phi(I) = \{x \in G : \exists\, x_i, y_i, \in I,\ x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}.$$

Then $\phi(I)$ is an o-ideal of (G, u). The mapping ϕ defines a one-to-one mapping preserving the set-theoretical inclusion. The inverse mapping ψ is given by

$$\psi(K) := K \cap [0, u],\ K \in \mathcal{I}(G, u).$$

The restriction of ϕ to $\mathcal{P}(E)$ or $\mathcal{M}(E)$ gives a bijection between $\mathcal{M}(E)$ and $\mathcal{M}(G)$, and $\mathcal{P}(E)$ and $\mathcal{P}(G)$, respectively, which preserves the set-theoretical inclusion.

In addition, if K is a prime o-ideal of (G, u), then $\gamma(K) := K \cap E$ is a prime ideal of E, and if P is a prime ideal of E, then $\phi(P)$ is a prime o-ideal of (G, u).

Moreover, an analogous result to Proposition 3.1 holds.

Proposition 3.3. *A proper o-ideal P of a unital Abelian po-group (G, u) satisfying (*RDP*) is prime if and only if G/P is an antilattice.*

4. HOLLAND THEOREM AND VISUALIZATION OF EFFECT ALGEBRAS

In the present section, we give the main results concerning representation of effect algebras with (RDP) and of MV-algebras by a set of automorphisms of an antilattice or a linearly ordered set. Having such a representation, we are able to visualized effect algebras and better to describe them.

Let $(\Omega, \leq)$ be a nonvoid antilattice, and let $A(\Omega)$ be the set of all automorphisms $\alpha : \Omega \to \Omega$, which preserve the partial order $\leq$. Then $A(\Omega)$ can be converted into a po-group such that the group-addition is the composition of automorphisms, the order on $A(\Omega)$ is defined via $\alpha \leq \beta$ iff $(\omega)\alpha \leq (\omega)\beta$ for all $\omega \in \Omega$, and the neutral element is the identity function on Ω.

Holland (1963) proved the basic result that every ℓ-group can be embedded into the ℓ-group $A(\Omega)$ for some linearly ordered set Ω, and Glass (1972, Theorem 54) generalized this result for directed po-groups satisfying (RIP) showing that every such a po-group can be embedded into the po-group $A(\Omega)$ for some antilattice Ω.

We show that a similar result can be proved also for effect algebras proving that every pseudoeffect algebra E satisfying (RDP) can be represented as the set of automorphisms of an antilattice Ω. Such a representation enables to present effect algebras in a visualized form which can be useful more more precise investigation of effect algebras.

Theorem 4.1. *Every effect algebra E with (*RDP*) can be represented as an effect algebra of automorphisms from* $A(\Omega)$ *for some antilattice set* Ω *such that all finite infima and suprema existing in E are preserved.*

Proof: Without loss of generality, by Theorem 2.1, we can assume that $E = \Gamma(G, u)$, where (G, u) is an Abelian interpolation unital po-group. The proof will follows the following steps.

Step 1. Let P be a prime ideal of E. According to Theorem 3.2, $\phi(P)$ is a prime subgroup of G, and consider a mapping $\phi_P : E \to A(\Omega_P)$, where $\Omega_P = G/\phi(P)$, defined by $(x/\phi(P))\phi_P(a) := (x + a)/\phi(P)$, $a \in E (x \in G)$. Then, for $a, b \in E$, (i) $a \leq b$, implies $\phi_P(a) \leq \phi_P(b)$, (ii) $\phi_P(a + b) = \phi_P(a) \circ \phi_P(b) = \phi_P(b + a) = \phi_P(b) \circ \phi_P(a)$, (iii) $\phi_P(a \wedge b) = \phi_P(a) \wedge \phi_P(b)$ if $a \wedge b$ is defined in E, (iv) $\phi_P(a \vee b) = \phi_P(a) \vee \phi_P(b)$ if $a \vee b$ is defined in E, and (v) $\{a \in E : \phi_P(a) = 0\} = \cap\{-x + \phi(P) + x : x \in G\} \cap E = P$.

Step 2. Let $g \in G$ and let $g \not\leq 0$ and set $U(g) := \{h \in G : h \geq g\}$, where $E = \Gamma(G, u)$. We denote by $A(g)$ an ideal of E that is maximal with respect to the property $U(g) \cap A(g) = \emptyset$. Since $0 \notin U(g)$, $A(g)$ exists due to the Zorn lemma. We assert $A(g)$ is a prime ideal of E. Let $I \cap J = A(g)$, where I and J are ideals of E. Assume (an absurdum hypothesis) $A(g)$ is a proper subset of I as well as of J. Take $a \in I \cap U(g)$ and $b \in J \cap U(g)$. We have $0, g \leq a, b$. By the interpolation holding in (G, u), there is an element $c \in G$ such that $0, g \leq c \leq a, b$. Since $0 \leq c \leq a$, we have $c \in E$, and $g \leq c \in I \cap J = A(g)$, which gives $c \in U(g) \cap A(g)$, a contradiction.

Step 3. We define the Cartesian product $E_0 = \prod\{A(\Omega)_g : g \in G, g \not\leq 0\}$ of the system of antilattices $\{A(\Omega_g)\}_g$, where $\Omega_g = G/C_g$ and $C_g = \phi(A(g))$, and we order E_0 by coordinates. Define a mapping $f : E \to E_0$ by $f(a) = \{\phi_g(a)\}_g (a \in E)$, where $\phi_g := \phi_{A(g)}$.

We claim that f is injective. Assume $f(a) = f(b)$. Then $(x + a)/C_g = (x + b)/C_g$ for all $x \in G$ and $g \not\leq 0$. In particular, for $x = 0$ that gives $a/C_g = b/C_g$. Hence, $a - b = c_g$ for some $c_g \in A_g(a - b$ is taken in the group $G)$, consequently, $a - b \in \cap_{g \not\leq 0} C_g = \{0\}$. This proves that f is an injective mapping from E onto $f(E) \subseteq E_0$.

Assume $f(a) \leq f(b)$. If $g = a - b \not\leq 0$. Then $(x + a)/C_g \leq (x + b)/C_g$ for all $x \in G$ and $g \not\leq 0$. Consequently, this holds also for $x = 0$, i.e., $a/C_g \leq b/C_g$, which means $a \leq c_g + b$. Hence, $a - b \leq c_g$, and $c_g \in A_{(a-b)} \cap U(a - b)$, a contradiction according to Step 2. Therefore, $f(E)$ can be converted into an effect algebra, i.e., $(f(E); \circ, f(0), f(1))$ is an effect algebra isomorphic with E, where $\circ$ is the composition of automorphisms defined by coordinates.

According to Step 1, f preserves all existing finite infima and suprema existing in E.

Step 4. Totally order the nonnegative elements of G, say $\{g_t : t \in T\}$, where T is a linearly ordered set. Set $\Omega_t := G/C_{gt}$, and without loss of generality we can assume $\Omega_s \cap \Omega_t = \emptyset$ for all $s, t \in T$ such that $s \neq t$. Let $\Omega = \cup_{t \in T} \Omega_t$, and define a partial order $\preceq$ on Ω by $\omega_1 \preceq \omega_2$ iff $\omega_2 \in \Omega_s$ and $\omega_1 \in \Omega_t$ and $s < t$ or $s = t$ and $\omega_1 \leq \omega_2$ in Ω_s. Then Ω is an antilattice with respect to $\preceq$.

Define a mapping $f_0 : E \to A(\Omega)$ defined via: let $\omega \in \Omega$, and $\omega \in \Omega_t$ for a unique $t \in T$. Let $(\omega) f_0(a) = (\omega)(\phi_{gt})(a) \in \Omega_t$, where ϕ_{gt} is defined in Step 1 and Step 3. Hence, if $a \in E$, then $f(a) \mid \Omega_t$ maps Ω_t onto Ω_t for all $t \in T$. Similarly as in Step 3, f_0 is injective from E onto $f_0(E)$, and $f_0(E)$ is an effect algebra of automorphisms of Ω (indeed, f_0 practically coincides with the function f defined in Step 3), which finishes the proof. □

If M is an MV-algebra, then its visualization has the following form.

Corollary 4.2. *Every MV-algebra M can be represented as an MV-algebra of automorphisms from $A(\Omega)$ for some linearly ordered set Ω.*

Proof: Since every MV-algebra is a distributive lattice, and as an effect algebra it fulfils (RDP), an ideal of an MV-algebra M (considered as an effect algebra) is prime iff M/P is a linearly ordered set. Consequently, $M = \Gamma(G, u)$ for some Abelian unital ℓ-group (G, u). Hence, the set Ω from the proof of Theorem 4.1 is linearly ordered, which by Theorem 4.1 gives the assertion in question. □

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